An Introduction to Unbounded Operators Part - 1

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Outline of the talk

- Motivation for solving operator equations corresponding to unbounded operators
- Introduction with some algebraic operations on operators
- Definitions and examples of closable operators, closures, closed operators and non-closable operators
- In the language of graph of operators, we see definitions of closable operators and closed operators
- Few results including closed graph theorem
- Solution of a simple boundary value problem - from the point of view of operator theory (the second derivative as an operator)
- Some stability results for solving an operator equation (which comes from a boundary value problem)

Notations

- H, an infinite-dimensional Hilbert space (not necessarily separable) over the field $\mathbb{K}$ of real or complex scalars.
- $B(H)$, the space of all bounded linear operators on H
- $R(T)$, the range of T
- $N(T)$, the null space of T
- I, the identity operator
- spaces : $c_{00}, \ell_2, L_2[0,1], C[0,1], C^1[0,1]$

Motivation

Consider the differential map

$$x \mapsto x'$$

on a suitable subspace of $C[0,1]$ with the sup norm $\|.\|_\infty$.

Observations :

1. The map is not defined everywhere on $C[0,1]$. For instance, the function defined by $x(t) = |t - \frac{1}{2}|$ is not differentiable.
2. It has a dense domain. The domain of it is a subspace consisting of all differentiable functions whose derivatives are continuous on $[0,1]$, denoted by $C^1[0,1]$.
3. It is a linear operator defined on $C^1[0,1]$.
4. It is not continuous.

We denote the map as $T : C^1[0,1] \to C[0,1]$ defined by $Tx = x'$.

Introduction

- This type of operators arise in boundary value problems and they are not everywhere defined on Hilbert spaces.
- Moreover, they are not continuous on their domain of definition, i.e., they are unbounded operators .
- The theory of unbounded operators developed in the late 1920s and early 1930s as part of developing a rigorous mathematical framework for quantum mechanics.
- They are called unbounded observables in quantum mechanics.

Introduction

The theory's development is due to John von Neumann and Marshall Stone. John von Neumann introduced graphs to analyze unbounded operators in 1936.

Elementary algebraic operations of addition and multiplication on operators

Specification of a domain is an essential part of the definition of an unbounded operator.

Let T and S be linear operators and let α be a scalar. Then the operators $T+S, TS, \alpha T$ and T^{-1} are defined as follows :

1. $D(T+S) = D(T) \cap D(S), \quad (T+S)x = Tx + Sx.$
   It may be possible that even though S and T are densely defined $D(S+T)$ may be a trivial subspace.
2. $D(TS) = \Big\{x \in D(S) : Sx \in D(T)\Big\}, \quad (TS)x = T(Sx).$
3. If $\alpha = 0$, then $\alpha T = 0$, otherwise
   $D(\alpha T) = D(T)$ and $(\alpha T)x = \alpha(Tx)$.
4. If T is injective, then $D(T^{-1}) = R(T)$ and $T^{-1}y = x$ if $y = Tx$.

Extension / Restriction of an operator

- Two operators T and S are **equal** if $D(T) = D(S)$ and $Tx = Sx$ for all $x \in D(T)$.
- S is an **extension** of T if $D(T) \subset D(S)$ and $Tx = Sx$ for all $x \in D(T)$. Here T is called a **restriction** of S and it is denoted by $T \subset S$.

Example 1.

Let $T_1 : C^1[0,1] \to C[0,1]$ by $T_1 x = x'$
and $T_2 : C^2[0,1] \to C[0,1]$ by $T_2 x = x'$. Here T_2 is a restriction of T_1.

Notation : In dealing with unbounded operators, one can express relations using vectors or the operators themselves. For example, the expression $STx = Sx$ $(x \in D(T))$ is same as $ST \subset S$.

- **Associative Laws :**
  $(AB)C = A(BC)$ and $(A+B)+C = A+(B+C)$.
- **Distributive Laws :**
  $(A+B)C = AC + BC$ and $A(B+C) \subset AB + AC$.
- **Commutative Law :**
  In general, $D(ST) \neq D(TS)$, and hence $ST \neq TS$.
- **Inverse :**
  In general, $TT^{-1} \neq TT^{-1}$.

It often happens that although T is unbounded, it has bounded inverse. In this case $T^{-1}T \subset TT^{-1} = I$, where I is the identity operator.

Note that $ST = TS$ does not imply $S^{-1}T^{-1} = T^{-1}S^{-1}$.

Invertible operator

Definition 2.

An unbounded operator $T : D(T) \to H$ *is said to be* **invertible** *if there exists an everywhere defined bounded operator* S *such that* $ST \subset TS = I$.

This definition is adopted in the books written by J.B. Conway and I. Gohberg et. al. Of course, in some textbooks, they do not assume the inverse defined everywhere as in the book by Kato.

Some results fail to hold if Kato's definition for invertibility is assumed : Let S and T be unbounded invertible normal operators. If $ST = TS$, then TS is normal (will be proved later).

Closable operators / closure / closed operators

Let $T : D(T) \to H$ be a linear operator.

Suppose T is bounded.

1. We have continuity.
2. We have the bounded extension theorem to extend T by continuity from $D(T)$ to a linear operator defined on $\overline{D(T)}$.

Moreover, the operator defined on $\overline{D(T)}$ can be further extended to the whole space H because $\overline{D(T)}$ is complemented in H.

Closable operators / closure / closed operators

Suppose T is unbounded.

Then the following two phenomena obviously occur.

1. We loose continuity.
2. We loose bounded extension theorem.

Both phenomena lead in a natural way to the notion of closable / closed operators.

Closable operators

Consider first, the loss of continuity : Given $D(T) \ni x_n \to x$.

It may happen that

1. limit of $\{Tx_n\}$ may not exist ;
2. Even the limit of $\{Tx_n\}$ exists, it may not be the same with some other sequence $D(T) \ni \tilde{x_n} \to x$;
3. For any sequence $D(T) \ni x_n \to x$, even all limits of $\{Tx_n\}$ are the same, say y, which may not be same with Tx.

Any of these possibilities prevents T to be extended "by continuity" to all the limit points of $D(T)$, i.e., to $\overline{D(T)}$.

Closable operators

Among the three possibilities, the second case is not "not so bad" circumstance if the following happens:

Not along all sequences $D(T) \ni x_n \to x \in H$, $\{Tx_n\}$ has a limit. Nevertheless for all sequences in $D(T)$ converging to x along which $\{Tx_n\}$ has a limit, this limit is unique.

In other words, if $D(T) \ni x_n \to x \in H$ and $D(T) \ni \tilde{x_n} \to x \in H$ such that $\lim Tx_n$ and $\lim T\tilde{x_n}$ exist, then $\lim Tx_n = \lim T\tilde{x_n}$. This situation (circumstance) leads to the definition of closable.

Closable operators

Definition 3.

Let $T : D(T) \to H$ be a linear operator.
The operator T is **closable** *if given any limit point x of $D(T)$, for all sequences $\{x_n\}$ converging to x and $\{Tx_n\}$ has a limit, such a limit is the same.*

It is also possible for a closable operator to have many closed extensions. Its minimal closed extension $\overline{T}$ is called its **closure**.
That is, every closed extension of T is also an extension of $\overline{T}$.

Closure

If T is closable, there is a natural candidate for its closure. How to find it?

Let $T : D(T) \to H$ be a linear operator. If T is closable, the **closure** of T is the operator $\overline{T}$ whose domain and action are

$$D(\overline{T}) = \Big\{ x \in H : \text{ there exists a sequence } \{x_n\} \text{ from } D(T) \text{ such that } x_n \to x \text{ and for which } \{Tx_n\} \text{ is also convergent} \Big\}.$$

Here uniqueness of y comes from the definition of closability of T, which will make the following operator well-defined.

Define $\overline{T}x = y$ for $x \in D(\overline{T})$.

Thus $T \subset \overline{T}$ for every closable T (Exercise).
In particular, $\overline{T}$ is the smallest closed extension of T (Exercise).

There are pitfalls for the unwary : $D(\overline{T})$ may be different from $\overline{D(T)}$.

It may happen that there is a limit point $x \in H$ of $\{x_n\}$ in $D(T)$, the sequence $\{Tx_n\}$ may not converge. In this case, $\overline{T}$ is not defined on x, because

$$\overline{T}\left(\lim_{n\to\infty} x_n\right) = \overline{T}x := \lim_{n\to\infty} Tx_n.$$

Definition 4.

Let $T : D(T) \to H$ be a linear operator. The operator T is **closed** *if $T = \overline{T}$.*

Closed Operators

Consider the following three facts for a linear operator T on H :

(i) $D(T) \ni x_n \to x \in H$.

(ii) $Tx_n \to y \in H$.

(iii) $Tx = y$.

Then

- T is closed if $(i) + (ii) \implies (iii)$.
- T is bounded (everwhere defined on H or defined on a closed subspace of H) if $(i) \implies (ii) + (iii)$.

Thus every bounded operator defined on the whole space H or has a closed subspace as domain, is closed.

Non-closabale operators

Non-closable operators are "too pathological."

The spectrum of unbounded operators, even closed ones, can be any closed set including $\emptyset$ and $\mathbb{C}$.

The domain of defintion plays an important role. In general, the larger the domain is, the larger the spectrum is.

Graph of an operator

Let $T : D(T) \to H$ be a linear operator. The **graph** of T defined by

$$G(T) = \Big\{(x, Tx) : x \in D(T)\Big\} \subseteq H \times H.$$

is a subspace of $H \times H$.

Note that $H \times H$ is naturally equipped with the inner product

$$\Big\langle (x, y), (x', y') \Big\rangle_{H \times H} = \langle x, x' \rangle_H + \langle y, y' \rangle_H$$

which makes it a Hilbert space.

Definitions of closable/closed operators using graphs

- T is **closable** if $\overline{G(T)} = G(S)$, for some linear operator S.
- T is **closed** if $\overline{G(T)} = G(T)$.

Proposition 5 (Exercise).

Let $T : D(T) \to H$ be a linear operator.

1. *T is closable if and only if the following holds : If $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \to 0$, and the sequence $\{Tx_n\}$ in H is convergent, then we have $\lim Tx_n = 0$.*
2. *T is closed if and only if the following holds : If $\{x_n\}$ is a sequence in $D(T)$ that is convergent in H and the sequence $\{Tx_n\}$ is convergent in H, then we have*

$$\lim x_n \in D(T) \text{ and } T(\lim x_n) = \lim Tx_n.$$

Graph of an operator between Hilbert spaces

Let H_1 and H_2 be Hilbert spaces and let $T : D(T) \subseteq H_1 \to H_2$ be a linear operator. The **graph** of T defined by

$$G(T) = \Big\{(x, Tx) : x \in D(T)\Big\} \subseteq H_1 \times H_2.$$

is a subspace of $H_1 \times H_2$.
Note that $H_1 \times H_2$ is naturally equipped with the inner product

$$\Big\langle (x, y), (x', y') \Big\rangle_{H_1 \times H_2} = \langle x, x' \rangle_{H_1} + \langle y, y' \rangle_{H_2}$$

which makes it a Hilbert space.

- T is **closable** if $\overline{G(T)} = G(S)$, for some linear operator S.
- T is **closed** if $\overline{G(T)} = G(T)$.

Graph of an operator between Banach spaces

Let X and Y be Banach spaces and let $T : D(T) \subseteq X \to Y$ be a linear operator. The **graph** of T defined by

$$G(T) = \Big\{(x, Tx) : x \in D(T)\Big\} \subseteq X \times Y.$$

is a subspace of $X \times Y$.

Note that $X \times Y$ is naturally equipped with the norm

$$\|(x, y)\|_{X \times Y} = \Big\{\|x\|_X^2 + \|y\|_Y^2\Big\}^{1/2}$$

which makes it a Banach space.

- T is **closable** if $\overline{G(T)} = G(S)$, for some linear operator S.
- T is **closed** if $\overline{G(T)} = G(T)$.

Example on Function Spaces

Example 6 (closed operator).

Let $C^1[0,1] \to C[0,1]$ *(with the sup norm) be defined by*

$$Tx = x'.$$

It is a known theorem from calculus that if

$$x_n(t) \to x(t), \quad x_n'(t) \to y(t)$$

uniformly in $[0,1]$*, then* $x(t)$ *is continuously differentiable and* $x'(t) = y(t)$*.*

Hence T *is closed.*

Example on Lebesgue Spaces

Example 7 (closed operator).

Consider the operator defined by $Tx = x'$ *in* $L_2[0,1]$ *with the domain*

$$D(T) = \Big\{x \in L_2[0,1] : x \text{ is absolutely continuous, } x' \in L_2[0,1], x(0) = 0\Big\}.$$

T *is a closed operator. Moreover, it has a bounded inverse.*

Example 8 (not closed but closable).

The operator defined by $Sx = x'$ *in* $L_2[0,1]$ *with the domain*

$$D(S) = \Big\{x \in C[0,1] : x' \in C[0,1], x(0) = 0\Big\}$$

is not closed but admits a closure. The closure of S *is the operator defined in the previous example (Example 7).*

Example on Sequence Spaces

Example 9 (not closed but closable).

Let $T : c_{00} \to \ell_2$ *be defined by* $T(x_n) = (nx_n)$*, where*

$$c_{00} = \Big\{ y \in \ell_2 : \exists N(y) \in \mathbb{N} \text{ such that } y_n = 0, \forall n \geq N(y) \Big\}.$$

The operator T *is not closed but it is closable.*

Example 10 (closure).

The operator S *defined by* $S(x_n) = (nx_n)$ *in* ℓ_2 *with the domain*

$$D(S) = \Big\{ (x_n) \in \ell_2 : (nx_n) \in \ell_2 \Big\}$$

is closed. Moreover, S *is the closure of the closed operator defined in the previous example (Example 9).*

Example 11 (non-closable operator).

Consider $T : L_2[0,1] \to L_2[0,1]$ *with* $Tx(t) = x(0).t$ *and* $D(T) = C[0,1]$.

This operator does not admit a closure. Indeed take

$$x_n(t) = \begin{cases} 1 - nt & 0 \leq t \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq t \leq 1 \end{cases}$$

we get $Tx_n(t) = t$ *but* $x_n(t) \to 0$ *in* $L_2[0,1]$.

Example on Sequence Spaces

Example 12 (non-closable operator).

Let $H = \ell_2$ *and* $D = \Big\{ y \in \ell_2 : \exists N(y) \in \mathbb{N} \text{ such that } y_n = 0, \forall n \geq N(y) \Big\}$.

Fix a vector $x_0 \in \ell_2 \setminus D$ *and put* $D(T) = D + span\{x_0\}$, *define*

$$T(y + \alpha x_0) = \alpha x_0, \quad y \in D, \alpha \in \mathbb{K}.$$

Then T *is a densely defined linear operator that is not closable.*

Exercise 13 (non-closable operator).

Let $\{e_n\}$ be an orthonormal basis of a separable Hilbert space H.

Let $D = span\{e_n\}, x_0 \in H \setminus D$. Take $D(T) = D + span\{x_0\}$ and define $T : D(T) \to \ell_2$ by

$$T(y + \alpha x_0) = \alpha x_0 \quad \text{for } y \in D, \alpha \in \mathbb{K}.$$

Show that $(0, x_0) \in \overline{G(T)}$ and deduce that T is not closable.

Example on Sequence Spaces

Exercise 14 (non-closable operator).

Let $\{e_n\}$ be an orthonormal basis of a separable Hilbert space H.

Let $D = span\{e_n\}$. Define the operator T on H with $D(T) = D$ by

$$T\Big(\sum_{j=1}^{n} \alpha_j e_j\Big) = \sum_{j=1}^{n} \alpha_j e_1.$$

Show that T is not closable.

Exercise 15 (closed operator).

Set $t : [0,1] \to \mathbb{C}$ *by*

$$t(s) = \begin{cases} 1 & s = 0 \\ \frac{1}{\sqrt{t}} & 0 < s \leq 1 \end{cases}$$

and define the **maximal operator of multiplication** T **by** t *on* $L_2[0,1]$*. That is,*

$$Tx = t\ x, \quad \text{for } x \in D(T) = \left\{ x \in L_2[0,1] : t\ x \in L_2[0,1] \right\}$$

Show that T *is a densely defined closed operator.*

Exercise 16 (closed operator).

Let $H = L_2(\mathbb{R})$ *with the inner product*

$$D(T) = \left\{x \in H : \int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt < \infty\right\}.$$

Define T *as* $(Tx)(t) = t\ u(t)$ *for* $x \in D(T)$.
Show that T *is unbounded and closed.*

Exercise 17 (closed operator).

Consider the following two operators defined by

$$Tx(t) = e^{2t}x(t) \quad \textit{and} \quad Sx(t) = (e^{-t} + 1)x(t)$$

on their respective domains

$$\begin{aligned} D(T) &= \{x \in L_2(\mathbb{R}) : e^{2t}x \in L_2(\mathbb{R})\} \\ D(S) &= \{x \in L_2(\mathbb{R}) : e^{-2t}x, e^{-t}x \in L_2(\mathbb{R})\}. \end{aligned}$$

Show that

1. T *is closed.*
2. S *is not closed since it has a closure* $\overline{S}$ *defined by* $Sx(t) = (e^{-t} + 1)x(t)$ *on* $D(S) = \{x \in L_2(\mathbb{R}) : e^{t}x \in L_2(\mathbb{R})\}$.

Some Results

Theorem 18.

Let $T : D(T) \to H$ be a linear operator.

1. *If T is closed, then $N(T)$ (the null space of T) is closed.*
2. *If T is injective, then T is closed if and only if T^{-1} is closed.*

Theorem 19.

Let $T : D(T) \to H$ *be a linear operator. On* $D(T)$ *by*

$$\langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle, \quad \|x\|_T = \left\{\|x\|^2 + \|Tx\|^2\right\}^{1/2}$$

an inner product and the corresponding norm (T*-norm or graph norm) are defined.*

T *is closed if and only if* $\Big(D(T), \langle ., .\rangle_T\Big)$ *is a Hilbert space.*

Some Results

Theorem 20.

Let $T : D(T) \subseteq X \to Y$ be a bounded linear operator from a normed space X to a Banach space Y. Then there exists a unique bounded extension S of T such that $D(S) = \overline{D(T)}$. We have $\|S\| = \|T\|$.

Theorem 21.

Every bounded operator is closable. A bounded operator T is closed if and only if $D(T)$ is closed. If T is bounded, then we have $D(\overline{T}) = \overline{D(T)}$; the closure $\overline{T}$ is the bounded extension of T onto $\overline{D(T)}$, constructed in Theorem 20.

Exercise 22.

Every finite-rank closable operartor is bounded.

Some Results

Theorem 23.

Let T be closable and injective.

The operator T^{-1} is closable if and only if $\overline{T}$ is injective. We then have $\overline{T^{-1}} = \overline{T}^{-1}$. If $\overline{T}^{-1}$ is continuous, then we have $R(\overline{T}) = \overline{R(T)}$.

Some Results

Theorem 24 (Closed Graph Theorem).

Let $T : D(T) \to H$ be a linear operator. Then the following statements are equivalent :

1. *T is closed and $D(T)$ is closed,*
2. *T is bounded and $D(T)$ is closed,*
3. *T is bounded and closed.*

Applying the closed graph theorem, the following observations are made for unbounded operators.

- The domain of an **unbounded** closed operator is a proper subspace of H.
- If T is **unbounded**, it is never true that $D(\overline{T}) = \overline{D(T)}$. [Hint : if it were, $\overline{T}$ and hence also T would be bounded.]
- Even if T is a densely defined **unbounded** operator, the domain of $\overline{T}$ is not the whole space H.

Theorem 25.

Let $T : D(T) \to H$ be an injective linear operator such that $R(T) = H$.

The operator T is closed if and only if $T^{-1} \in B(H)$ (bounded on H).

Stability Result : If it is known that the equation $Tx = y$ has exactly one solution for every $y \in H$ and if T is closed, then the solution depends continuously on y, by Theorem 25 (as T^{-1} is bounded on H).

The next two results provide useful criteria for the existence of a bounded inverse.

Some Results

Theorem 26.

Let $T : D(T) \to H$ be linear. Then the following hold :

1. *Suppose that there is an $m > 0$ such that*

$$\|Tx\| \geq m\|x\| \quad \text{for all } x \in D(T).$$

   Then T is closed if and only if $R(T)$ is closed.

2. *Assume that T is closed. Then $T^{-1} \in B(H)$ if and only if $R(T)$ is dense in H and there is an $m > 0$ such that*

$$\|Tx\| \geq m\|x\| \quad \text{for all } x \in D(T).$$

An example of boundary value problem

Consider the differential equation

$$\{p(x)f'(x)\}' + q(x)f(x) = g(x) \qquad 0 \leq x \leq 1 \tag{1}$$

together with the boundary conditions $f(0) = f(1) = 0$.

If p, q are smooth and $p(x) \neq 0$ for $x \in [0,1]$, the equation may be written in the form $Tf = g$ where T is a closed operator from $L_2[0,1]$ into itself.

If (1) has exactly one solution for every $g \in L_2[0,1]$, then the solution depends continuously on the right side g, when (1) arises from a specific problem, this is the result usually to be expected on physical grounds.

Closed Ranges

Theorem 27 (bounded case).

Let $T \in B(H)$. Then $R(T)$ is closed iff $T|_{N(T)^\perp}$ is bounded from below, i.e., there is some $m > 0$ such that $\|Tx\| \geq m\|x\|$, for all $x \in N(T)^\perp$.

Theorem 28 (closed operator case).

Let $T : D(T) \to H$ be a closed operator. Then $R(T)$ is closed iff $T|_{D(T)\cap N(T)^\perp}$ is bounded from below, i.e., there is some $m > 0$ such that $\|Tx\| \geq m\|x\|$, for all $x \in D(T) \cap N(T)^\perp$.

The subspace $C(T) := D(T) \cap N(T)^\perp$ is called the **carrier** of T.
In fact, $D(T) = N(T) \oplus^\perp C(T)$ (orthogonal direct sum).

Some Results

Theorem 29.

Let $T : D(T) \to H$ be an injective closed linear operator. Assume that there is a linear operator S from H into H with $R(S) \subset D(T)$ and with domain dense in H, and suppose that $TS \subset I$ $\Big(TSx = x, \forall x \in D(S)\Big)$.

If S is bounded on its domain, then $T^{-1} \in B(H)$ and $T^{-1} = \overline{S}$.

Simple Boundary Value Problem

Consider the simple boundary value problem

$$\begin{aligned} -y''(x) &= f(x) \\ y(0) &= y(1) = 0, \end{aligned}$$

where f is a function in $L_2[0,1]$.

To find a solution, we integrate twice and obtain

$$y(x) = -\int_0^x \int_0^t f(s)\, ds\, dt + c_1 x + c_2, \tag{2}$$

where

$$c_2 = 0 \text{ and } c_1 = \int_0^1 \int_0^t f(s)\, ds\, dt.$$

Simple Boundary Value Problem

Interchanging the order of integration yields

$$\begin{aligned} y(x) &= -\int_0^x \int_s^x f(s)\ dt\ ds + x \int_0^1 \int_s^1 f(s)\ dt\ ds \\ &= \int_0^x (s-x)\ f(s)\ ds + \int_0^1 x(1-s)\ f(s)\ ds. \end{aligned}$$

Hence

$$y(x) = \int_0^1 g(x,s)\ f(s)\ ds,$$

where

$$g(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1. \end{cases}$$

The Second Derivative as an Operator

Conversely, if y is given by

$$y(x) = \int_0^1 g(x,s)\, f(s)\, ds,$$

a straightforward computation verifies that y satisfies

$$-y''(x) = f(x) \text{ and } y(0) = y(1) = 0$$

almost everywhere.

The function g is called the **Green's function** corresponding to the boundary value problem.

The Second Derivative as an Operator

Let us consider the above result from the point of view of operator theory. We want to express the differentiable expression $-y''$ with the boundary conditions $y(0) = y(1) = 0$ as a linear operator.

The action of the operator is clear. However, we must define its domain.

To do this, we note that (2) implies that the derivative y' is an indefinite integral or, equivalently, y' is absolutely continuous.

An important property of absolutely continuous functions, is that the usual "integration by parts" formula holds for the integral of fg, where f is absolutely continuous and g is Lebesgue integrable.

The Second Derivative as an Operator

Let the domain $D(T)$ of T be the subspace of $L_2[0,1]$ consisting of those complex-valued functions y which satisfy $y(0) = y(1) = 0$, have first order derivatives which are absolutely continuous on $[0,1]$ and have second order derivatives which are in $L_2[0,1]$.

Note that $y''(x)$ exists for almost every x since y' is absolutely continuous. Define $Ty = -y''$.

Exercise : Show that T is an injective closed linear operator on $L_2[0,1]$.

General solution of this problem will be discussed in the next lecture.

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